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# Natural convection in vertical porous enclosures with internal heat generation

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**Abstract**—Low- and high-Rayleigh number convections induced by internal heat generation in a vertical porous cylinder are studied in different ways according to their flow structures. At low Rayleigh numbers, a first-order partial Galerkin scheme is proved to be effective. Convection at high Rayleigh numbers is characterized by a homogeneous upward flow in the central part of the cylinder and a very thin downward boundary layer on the cooling wall, with the effect of the curvature of the boundary being negligible. By introducing a characteristic length in the same form as the hydraulic radius, the results can be applied to any other enclosures with non-circular boundaries. Comparison of results for a cylinder with those for a rectangular enclosure strongly supports the idea.

## 1. INTRODUCTION

NATURAL convection in porous enclosures with internal heat generation has received more and more attention owing to its possible applications in engineering problems, such as heat removal from radioactive waste materials, storage of agricultural products, exothermic chemical reaction, etc. Haajizadeh *et al.* [1] have theoretically studied natural convection inside a rectangular porous enclosure with uniform heat generation. Vasseur *et al.* [2] have presented numerical results on natural convection in an internally heated horizontal porous annulus. Stewart and Dona [3] were concerned with a situation similar to that in a cylindrical food-storage bin which was modellized with a heat generating vertical porous cylinder, adiabatic at the bottom and cooled at the side wall and the top boundaries.

In the present study, we deal with natural convection in a vertical porous cylinder with uniform internal heat generation and side-wall cooling by the partial Galerkin method and the finite difference method. In the low Rayleigh number region, an effective partial Galerkin scheme is developed. Its first-order approximation, valid at low Rayleigh numbers by comparison with the finite difference scheme, gives a better physical explanation of the problem. In the high Rayleigh number region, emphasis is placed on the structures of the stratification at the interior region and boundary-layer flow along the vertical wall of the cylinder. The flow in a cylinder is also compared with that in a vertical rectangular enclosure to examine whether the curvature of the cooled wall has any effect on the flow structure and heat transfer at high Reynolds numbers when boundary-layer flow is dominant.

## 2. MATHEMATICAL MODEL

The geometry of the problem considered is shown in Fig. 1. Owing to the symmetry of the boundary conditions and the gravitational force, steady-state flows in the cylinder are assumed to be two-dimensional and symmetrical about the central vertical axis of the cylinder. The governing equations, i.e. Darcy's equation and the equation of the conservation of energy, are given in their dimensionless form [4] as follows:

$$\left(\nabla^2 - \frac{2}{r}\frac{\partial}{\partial r}\right)\psi = Ra\left(r\frac{\partial T}{\partial r}\right) \tag{1}$$

$$u\frac{\partial T}{\partial r} + v\frac{\partial T}{\partial z} = \nabla^2 T + f \tag{2}$$

where $T$ stands for temperature, $u$ and $v$ velocities in the $r$- and $z$-directions, respectively, $f$ the distribution function of heat generation rates, and $\nabla^2$ the Laplacian operator in the cylindrical coordinate system. Stream function $\psi$ is defined as

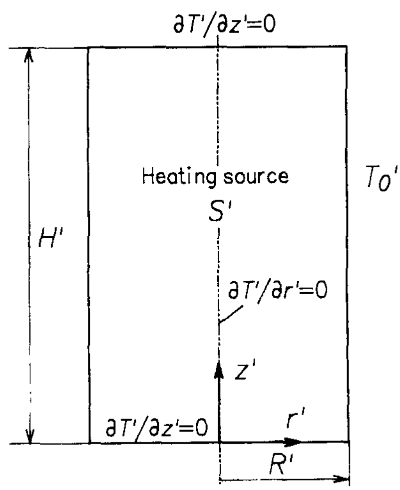


FIG. 1. Flow geometry and coordinate system.

## NOMENCLATURE

| | |
|---|---|
| $A_h$ | area of a horizontal section |
| $A_s$ | area of vertical cooling surface |
| $c$ | specific heat capacity |
| $f$ | distribution function of heat generation |
| $f_1, f_2$ | functions defined in partial Galerkin scheme for $\psi$ and $T$, respectively |
| $g$ | gravitational acceleration |
| $H$ | aspect ratio, $H'/R'$ |
| $H'$ | height |
| $k$ | effective thermal conductivity |
| $L'$ | width of a vertical rectangular enclosure |
| $M, N$ | truncating numbers of expansion series |
| $Nu$ | local Nusselt number |
| $p$ | pressure, $p'k/(\mu\alpha)$ |
| $R'$ | radius of a cylinder |
| $r$ | radial distance, $r'/R'$ |
| $Ra$ | Rayleigh number, $g\beta S\kappa R'^3/(\alpha\nu k)$ |
| $R_e$ | equivalent radius defined in equation (10) |
| $s$ | total length of cooling boundaries around a vertical section |
| $S'$ | heat generation rate |
| $T$ | temperature, $(T'-T'_0)k/(S'R'^2)$ |
| $T_c$ | temperature at pure-conduction state |
| $T_{max}$ | maximum temperature |
| $u$ | radial velocity component, $u'R'/\alpha$ |
| $v$ | velocity vector, $[u, v]$ |
| $v$ | vertical velocity component, $v'R'/\alpha$ |
| $z$ | vertical distance, $z'/R'$. |

Greek symbols

| | |
|---|---|
| $\alpha$ | equivalent thermal diffusivity |
| $\beta$ | thermal expansion coefficient |
| $\theta$ | temperature defined by $(T-T_c)$ |
| $\kappa$ | permeability |
| $\mu$ | dynamic viscosity |
| $\nu$ | kinematic viscosity |
| $\rho$ | fluid density |
| $\psi$ | stream function. |

Superscript

| | |
|---|---|
| $'$ | dimensional variable. |

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}; \quad v = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{3}$$

The boundary conditions on $\psi$ and $T$ are

$$\psi = 0 \quad \text{at all boundaries}$$

$$\frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{at } r = 0$$

$$T = 0 \quad \text{at } r = 1. \tag{4}$$

In the present paper, we only consider the situation of uniform heating, i.e. $f = 1$ in equation (2), and adiabatic top boundary. However, non-uniform heat generation and top-boundary cooling can have significant effects on heat transfer, particularly the maximum temperature: these results are reported elsewhere [4].

### 2.1. *Partial Galerkin scheme*

Stream function $\psi$ and temperature $T$ in equations (1) and (2) can be partially expanded as follows:

$$\psi = f_1(r)\sum_{n=1}^{N}\sin n\pi z$$

$$T = T_c + \varepsilon = \tfrac{1}{4}(1-r^2) + f_2(r)\sum_{m=1}^{M}\cos m\pi z \tag{5}$$

where $T_c = (1-r^2)/4$ is the temperature of pure-conduction solution and $\theta$ the deviation from it. It was proved that using the trigonometric functions in expanding solutions is quite efficient to the problem of natural convection in a porous medium [5, 6], and that the first-order approximation can bring about very simple and relatively accurate solutions at low Rayleigh numbers [5].

Although solutions can also be obtained for any higher truncation numbers $N$ and $M$, we focus our attention here on the first-order approximation, i.e. $N = 1$ and $M = 1$, which, as will be shown later, can give an interesting physical explanation to solutions at low Rayleigh numbers.

According to equation (5), the first-order or first-term approximation is

$$\psi = f_1(r)\sin\pi z$$

$$T = \tfrac{1}{4}(1-r^2) + f_2(r)\cos\pi z. \tag{6}$$

Substituting equation (6) into equations (1) and (2), multiplying the resulting equations by the trial functions $\sin\pi z$ and $\cos\pi z$ respectively and integrating them in the cylindrical space, the standard Galerkin procedure [5] thus leads to the following linear, ordinary differential equations:

$$f_1'' - \frac{2}{r}f_1' - \frac{\pi^2}{H^2}f_1 = -\frac{2}{\pi}r^2\,Ra \tag{7}$$

$$f_2'' + \frac{1}{r}f_2' - \frac{\pi^2}{H^2}f_2 = \frac{1}{2}\left(\frac{\pi}{M}\right)f_1 \tag{8}$$

with the boundary conditions being

$$f_1(0) = f_1(1) = 0$$

$$f_2'(0) = f_2(1) = 0. \tag{9}$$

It is interesting to note that by just ignoring higher-order harmonics without any assumptions, i.e. by only considering the first term of the trigonometric expansions of equation (5), we obtained the simplified equations similar to those linearized by some assumptions with clear physical meanings, namely:

(a) in Darcy's equation, equation (1), the buoyancy force is induced mainly by the temperature distribution of the pure conduction pattern $T_0$, the part contributed by $\theta$, or $Ra(r\partial\theta/\partial r)$, can be ignored;

(b) in the energy equation, equation (2), energy transferred by convective flow is mainly carried out by $u\partial T_0/\partial r$, the part related to $\theta$ can be ignored.

These assumptions result from the orthogonality of trial functions and implicitly put into the governing equations in the straight-forward process of turning equations (1) and (2) into equations (7) and (8), and the results will show that they are valid in the low Rayleigh number region. This first-order approximation which has an interesting physical explanation is possible only when there exists trial functions suitable to the physical problem [7]. Equations (7) and (8) are solved with the finite difference method in one dimension, and the results are compared with complete finite difference solutions to the original equations, equations (1) and (2).

### 2.2. *Finite difference scheme*

The scheme is based on the simple SOR method to solve equations (1) and (2) and has been described previously [4]. Release factors are chosen between 0.5 and 1.5. Non-uniform mesh grids varying from 21 to 81 in one direction are adopted according to different $Ra$ and $H$. Iteration is continued until the change in temperature at each nodal point is less than $10^{-6}$.

## 3. RESULTS AND DISCUSSION

### 3.1. *Results for low Rayleigh numbers*

Streamlines and temperature distributions obtained with the finite difference method are shown in Fig. 2

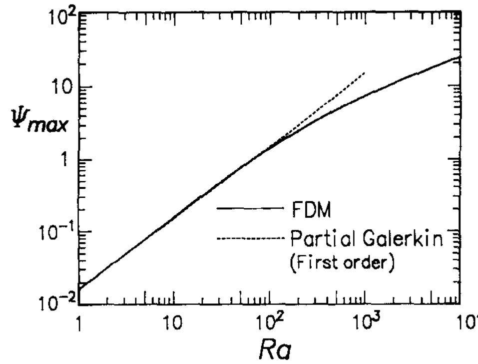

FIG. 3. Comparison between $\psi_{max}$ from the finite difference scheme and the first-order Galerkin scheme.

for a very low $Ra$. The small value of $\psi_{max}$ characterizes a very weak convective flow. The streamlines are almost symmetrical about the $z = H/2$ plane, confirming the validity of the first-order approximation of the partial Galerkin scheme which leads to such a symmetricity of $\psi$. The isotherms deviate only slightly from those of the pure-conduction state. Nevertheless, the central axis of the cylinder is no longer isothermal; owing to the occurrence of convection, the temperature along the axis increases with the height, reaching its maximum at the centre of the top boundary ($r = 0$, $z = H$). This can result in a maximum temperature higher than that calculated by the pure-conduction model, which would otherwise be conservative in estimating the maximum [4].

Figures 3 and 4 show values of $\psi_{max}$, or the circulating flow rates, and values of $T_{max}$ calculated by the finite difference scheme and the first-order Galerkin scheme, respectively. The results obtained by the two schemes agree very well with each other at Rayleigh numbers lower than 50, confirming the validities of this first-order approximation and the relevant assumptions. In Fig. 4 we can find that $T_{max}$ of the first-order approximation keeps increasing monotonically with $Ra$ and it deviates from results

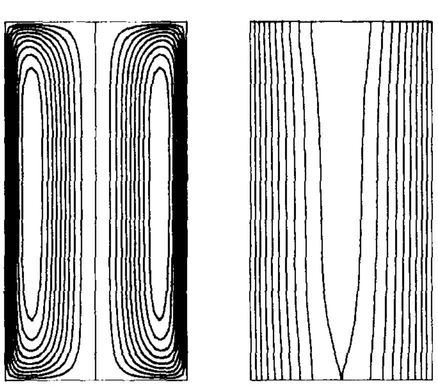

FIG. 2. Streamlines and isotherms for $Ra = 10$, $H = 4$, $\Delta\psi = \psi_{max}/10$, $\Delta T = T_{max}/10$, $\psi_{max} = 0.157$, $T_{max} = 0.263$.

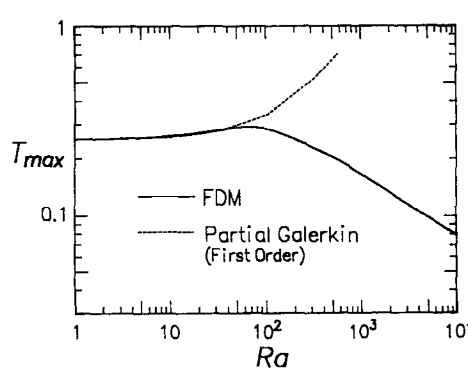

FIG. 4. Comparison between $T_{max}$ from the finite difference scheme and the first-order Galerkin scheme.

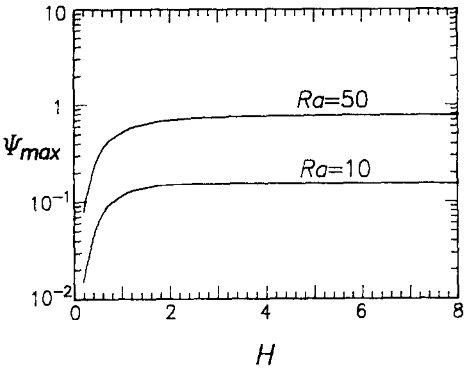


FIG. 5. Variations of $\psi_{max}$ with $H$ at low Rayleigh numbers.

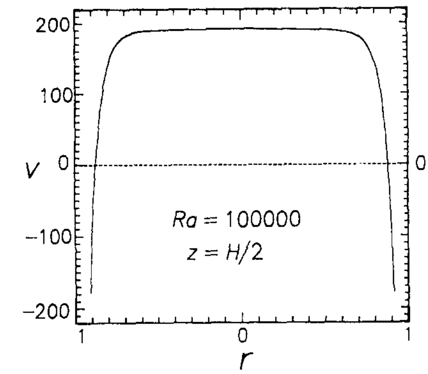


FIG. 7. Vertical velocity profile at $z = H/2$ for $Ra = 10^5$.

of the finite difference method exactly at the place where the latter begins to decrease with $Ra$. This is the result of ignoring other parts of the convective term except that corresponding to the pure-conduction temperature, or $u\partial T_0/\partial r$, so that the first-order approximation cannot describe the cooling effect of developed convection which lowers the temperature over the whole domain considered [4]. The range of $Ra$ in which the two results agree well is the same as that defined as the pseudo-conduction region, i.e. $Ra < 50$ [4]. We have proved that, at low $Ra$, increasing $H$ does not obviously change the flow and temperature fields near the top and bottom ends—its only effect seems to be that the parallel flow in the interior section is prolonged, and thus the values of $T_{max}$ and $\psi_{max}$ are invariable with $H$. As shown in Fig. 5, the $\psi_{max}$ for a certain $Ra$ reaches a constant as $H$ increases, and generally the $\psi_{max}$ for a higher $Ra$ converges to a constant at higher $H$. Up to $Ra = 50$, little change of $\psi_{max}$ can be observed at $H > 3$; in this range of $Ra$, therefore, the results shown in Figs. 3 and 4 for $H = 4$ can be applied to higher values of $H$.

### 3.2. *Results for high Rayleigh numbers*

Figure 6 shows results for a high Rayleigh number. We can find that most of the cylinder is occupied by the ascending hot flow, while the descending flow along the side wall becomes very thin and forms a boundary layer. A region of temperature stratification can be observed in the upper part of the cylinder, and it extends toward the lower part as $Ra$ increases; at very high $Ra$, it occupies most of the cylinder except at the bottom region or in the very vicinity of the side wall.

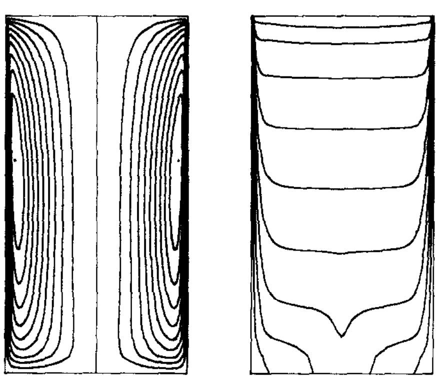

FIG. 6. Streamlines and isotherms for $Ra = 10^5$, $H = 4$, $\Delta\psi = \psi_{max}/10$, $\Delta T = T_{max}/10$, $\psi_{max} = 65.2$, $T_{max} = 0.037$.

Figure 7 shows the vertical velocity profile at $z = H/2$. The vertical velocity is a constant in most of the interior of the cylinder except near the cooling wall. As will be shown later, this flow pattern provides the possibility of applying the results of a cylinder to enclosures with non-circular boundaries.

The variations of $\psi_{max}$ and $T_{max}$ with $Ra$ are compared in Fig. 8 for different values of $H$. At low $Ra$ ($Ra < 50$), $\psi_{max}$ seems to be invariable with $H$. This is due to the dominance of conduction as mentioned in the discussions of Figs. 2 and 5. At higher $Ra$, or when convection becomes dominant, $\psi_{max}$ increases with increasing $H$. Since for a higher $H$, the path along which the ascending flow is heated is longer, the velocity as well as the circulation rate ($\psi_{max}$) becomes larger.

### 3.3. *Comparison between results for a cylinder and those for a rectangular enclosure*

Results in Figs. 6 and 7 bring in the following hypothesis. Since the upward flow at high $Ra$ is homo-

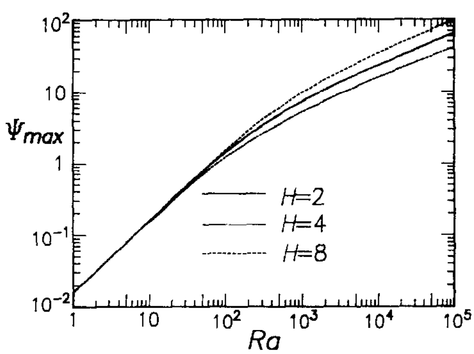


FIG. 8. $\psi_{max}$ for different $H$.

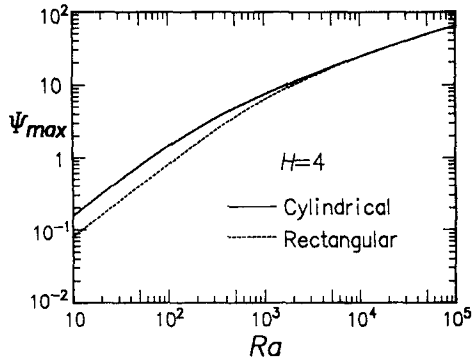


FIG. 9. Comparison between $\psi_{max}$ of a cylinder and a rectangular enclosure.

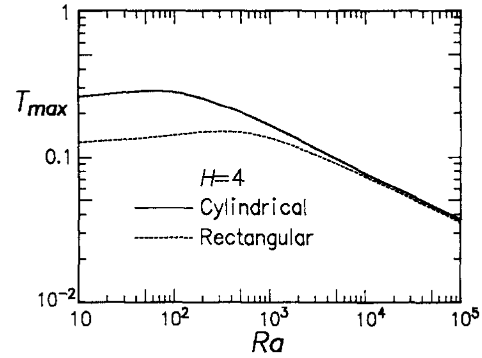


FIG. 10. Comparison between $T_{max}$ of a cylinder and a rectangular enclosure.

geneous in most of the cylinder, the downward boundary layer along the cooling wall is so thin that little effect of the curvature of the cooling boundary should be expected. In this case, the flow would be determined by only two factors—the area of the cooling boundary and the gross heat generation the boundary surrounds. In problems dealing with homogeneous heat generation, the two factors can be included by using the following ratio as the characteristic length, called in this paper the equivalent radius:

$$R_e = 2V_h/A_s = 2A_h/s \quad (10)$$

where $V_h$ stands for the volume of the heat-generating medium, $A_s$ the area of the cooling surface, $A_h$ the area of the horizontal section and $s$ the total length of the cooling boundary around the section. It is interesting to note that $R_e$ appears to be in the same form as the hydraulic radius of a forced flow in a pipe or channel, with $A_h$ corresponding to the area of a cross-section and $s$ the circumference of the section.

According to the definition in equation (10), the equivalent radius for a cylinder is its radius ($R'$) while that for a rectangular enclosure is the width ($L'$) between two vertical cooling walls. In equations (9)–(11), results for a cylinder of $H = 4$ are compared with those for a rectangular enclosure [8] of the same $H$. It should be noted that although the aspect ratios $H$ are equal to each other, the vertical geometrical projections are different; the projection of the cylinder is twice as wide as that of the rectangular enclosure owing to the difference in characteristic length.

Figure 9 shows that the values of $\psi_{max}$ are higher for the cylindrical than for the rectangular enclosures at low Rayleigh numbers. When the two cases have the same $S'$, $H'$ and equal equivalent radius $L' = R'$—consequently the same $Ra$ and $H$—convection is easier to develop in a cylinder than in a rectangular enclosure because the central vertical line around which the flow is most likely to occur is twice the distance from the cooling wall in a cylinder than in a rectangular enclosure when $L' = R'$. The same explanation can also be applied to values of $T_{max}$ at low Rayleigh numbers shown in Fig. 10. However, when temperature stratification and boundary-layer flow appear as a result of increasing $Ra$, the two lines converge to very close values, showing the similarity of the two flows. Figure 11 shows variations of local Nusselt numbers along the side wall for the two enclosures, with no meaningful difference being observed. All these results support the idea that high-Rayleigh number convection induced by internal heat generation in vertical porous enclosures can be generally treated by introducing the equivalent radius as the characteristic length, regardless of their real geometries.

## 4. CONCLUSION

Numerical results obtained with the finite difference method and the partial Galerkin method were presented on natural convection in a porous cylinder with internal heating and side-wall cooling, for an aspect ratio $H$ (height vs radius) of 0.2–8 and a Rayleigh number of $1$–$10^5$, with the low- and high-Rayleigh number convections being differently treated according to their flow structures.

At low-Rayleigh numbers ($Ra < 50$), streamlines are parallel in the central part of the cylinder except near the top and bottom ends. Convection in this $Ra$ range is mostly driven by the temperature distribution

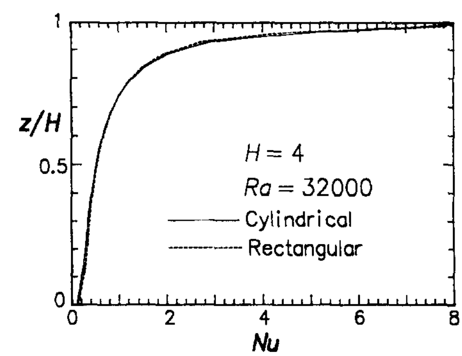


FIG. 11. Comparison between local Nusselt numbers of a cylinder and a rectangular enclosure.

of pure conduction $T_0$, with the effects of other terms in governing equations being negligible. A first-order partial Galerkin scheme serves as a very effective way to estimate values of $\psi_{max}$ and $T_{max}$.

At high-Rayleigh numbers ($Ra > 5000$), the convection is characterized by a homogeneous upward flow in most of the enclosure and a very thin downward boundary layer along the vertical cooling wall. Curvature of the vertical boundary seems to have little effect on flow structures. This convection can be generally treated by introducing a characteristic length called equivalent radius, or $A_h/s$, which is in the same form as the widely known hydraulic radius. By doing so, the results for a cylinder or a rectangular enclosure can be applied to any other vertical enclosures regardless of their geometries. Comparison was made between the results for a cylinder and those for a rectangular enclosure; their values of $\psi_{max}$, $T_{max}$ and local Nusselt numbers agreed very well with each other.

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## CONVECTION NATURELLE DANS DES ENCEINTES VERTICALES POREUSES AVEC GENERATION DE CHALEUR INTERNE


**Résumé**—La convection à faible ou fort nombre de Rayleigh induite par une génération de chaleur interne dans un cylindre poreux vertical est étudiée de différentes façons suivant les structures de l'écoulement. Aux petits nombres de Rayleigh, un schéma du premier ordre de Galerkin est prouvé efficace. La convection aux grands nombres de Rayleigh est caractérisée par un écoulement montant homogène dans la partie centrale du cylindre et une couche limite descendante très mince sur les parois froides, avec effet négligeable de la courbure de la couche limite. En introduisant une longueur caractéristique de même forme qu'un rayon hydraulique, les résultats peuvent être appliqués à d'autres enceintes à frontières non circulaires. Une comparaison des résultats sur le cylindre avec ceux sur une cavité rectangulaire appuie fortement cette idée.


## NATÜRLICHE KONVEKTION IN VERTIKALEN PORÖSEN STRUKTUREN MIT INNEREN WÄRMEQUELLEN


**Zusammenfassung**—Es wird die Konvektion in vertikalen porösen Zylindern bei niedriger und hoher Rayleigh-Zahl und entsprechend unterschiedlichen Strömungsformen untersucht. Die Konvektion wird durch innere Wärmequellen verursacht. Bei kleinen Rayleigh-Zahlen erweist sich ein partielles Galerkin-Verfahren erster Ordnung als geeignet. Konvektion bei hohen Rayleigh-Zahlen ist durch eine homogene Aufwärtsströmung im Kern des Zylinders und eine sehr dünne abwärtsströmende Grenzschicht an der gekühlten Wand gekennzeichnet, wobei sich der Einfluß der Krümmung der begrenzenden Wand als vernachlässigbar erweist. Die Einführung einer charakteristischen Länge ähnlich wie beim hydraulischen Radius führt dazu, daß die Ergebnisse auf Hohlräume mit nicht kreisförmigen Berandungen angewandt werden können. Ein Vergleich der Ergebnisse für einen Zylinder und für einen rechteckigen Hohlraum bestätigt dies.


## ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ В ВЕРТИКАЛЬНЫХ ОБЪЕМАХ ПОРИСТОГО МАТЕРИАЛА С ВНУТРЕННИМ ТЕПЛОВЫДЕЛЕНИЕМ


**Аннотация—Процессы конвекции в вертикальном пористом цилиндре, возникающие за счет внутреннего тепловыделения при низких и высоких числах Рэлея, исследуются различными методами в соответствии со структурой течения. В случае низких чисел Рэлея эффективной является частичная схема Галеркина первого порядка. Для конвекции при высоких числах Рэлея характерно существование однородного восходящего потока в центральной части цилиндра и очень тонкого опускного пограничного слоя на охлаждающей стенке, причем эффект кривизны границы пренебрежимо мал. При введении характерного размера в таком же виде, как и гидравлический радиус, полученные результаты могут применяться к другим полостям с некруглыми границами. Сравнение результатов для цилиндра и прямоугольной полости подтверждает справедливость предложенной гипотезы.**